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ORIGINAL PAPER

Second-order Wagner theory of wave impact

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Abstract The paper deals with the two-dimensional unsteady problem of the impact of a liquid parabola onto a rigid flat plate at a constant velocity. The liquid is assumed ideal and incompressible and its flow potential. The initial stage of the impact is the main concern in this study. The non-dimensional half-width of the contact region between the impacting liquid and the plate plays the role of a small parameter in this problem. The flow region is subdivided into four parts: (i) the main flow region, the dimension of which is of the order of the contact-region width, (ii) the jet-root region, where the curvature of the free surface is very high and the flow is strongly nonlinear, (iii) the jet region, where the flow is approximately one-dimensional, (iv) the far-field region, where the flow is approximately uniform at the initial stage of impact. A second-order solution in the main flow region has been derived and matched to the first-order inner solution in the jet-root region. The matching conditions provide an estimate of the dimension of the contact region for small time. Pressure distributions in both the main flow region and the inner region are derived. The accuracy of the obtained asymptotic formulae is estimated. The second-order hydrodynamic force acting on the plate is obtained and compared with available experimental data. A fairly good agreement is reported.

Keywords Flat plate · Hydrodynamic force · Second-order solution · Wagner theory · Wave impact

1 Introduction

The initial stage of wave impact onto a rigid and horizontal flat plate is studied. The wave profile close to the impact point is approximated by a parabolic contour. The liquid flow induced by the impact is two-dimensional and symmetric with respect to the vertical line, which is normal to the plate at the point of impact. The liquid is assumed ideal and incompressible. Gravity and surface-tension effects are not taken into account. The liquid flow is assumed potential. Before impact the liquid parabola moves up at constant velocity V and touches the plate at the impact instant, which is taken as the initial one, at a single point, which is taken as the origin of a Cartesian coordinate system (see Fig. 1). The shape of the liquid region before impact is characterized only by the radius of curvature R at the parabola top. We shall determine the initial asymptotics of the liquid flow, the pressure distribution along the wetted part of the plate, the

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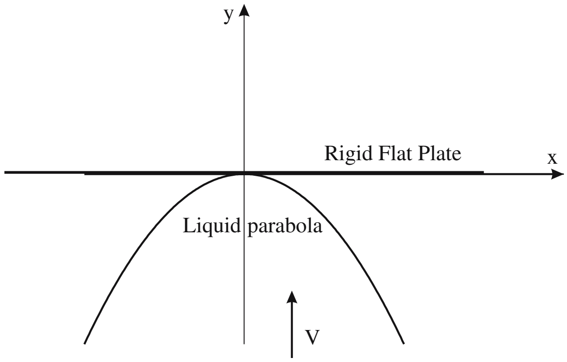

Fig. 1 Sketch of the flow just before the impact

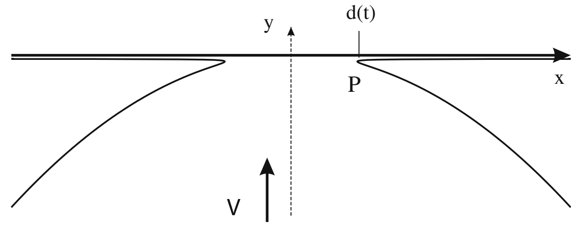

Fig. 2 The flow pattern during the initial stage of the liquid impact: $d(t)$ is the distance from the impact point to the turnover point P, where the tangent to the free surface is vertical; the jet length is infinite within the incompressible liquid model

dimension of the contact region between the impacting liquid and the rigid plate and the hydrodynamic force acting on the plate during the early stage, when the displacements of the liquid particles are much smaller than the length scale R of the problem. Due to the flow symmetry, only the right-hand side of the flow is considered in this paper.

The problem is formulated in non-dimensional terms with R being the length scale, R/V the time scale, VR the scale of the velocity potential and ρV^2 the scale of the hydrodynamic pressure. The sketch of the flow just before impact is shown in Fig. 1. The origin of the Cartesian coordinate system Oxy is taken at the impact point. The line $y = 0$ corresponds to the rigid flat plate. The line $x = 0$ is the symmetry line of the flow. At the impact instant, $t = 0$, the shape of the liquid free surface is described by the equation $y = -\frac{1}{2}x^2$ in non-dimensional variables. The flow pattern after the impact instant, $t > 0$, is shown in Fig. 2, where $c(t)$ is the half-length of the wetted part of the plate and $d(t)$ the distance from the impact point to the turnover point P at which the tangent to the free surface is vertical.

The turnover point plays a very special role in the impact problem. This is because during the early stage of the impact the jets are very thin and can be disregarded in calculations of the hydrodynamic force acting on the rigid body, as was suggested by Wagner [1]. Correspondingly, the wetted part of the plate (contact region between the liquid and the rigid surface) can be defined without taking account of the jets with its dimension being equal to $2d(t)$. In the present asymptotic analysis the quantity d is treated as a small parameter and the time $t(d)$ is considered an unknown function. Moreover, the problem is analyzed by using the stretched coordinates x/d and y/d, within which the horizontal coordinate of the turnover point P is fixed.

A peculiarity of the impact problem is connected with the fact that features of the flow in different parts of the flow region are different. This is why the flow domain is subdivided into four basic subdomains. The asymptotic solutions are obtained for each subdomain and matched to one another.

The main region of the disturbed flow is marked as the region I (see Fig. 3). The dimension of this region is of the order of $O(d)$ as $t \to 0$. This is the region where the flow is described by the Wagner solution in the leading order as $t \to 0$. In this region the free-surface slope is small, the actual position of the free surface can be approximated as $y = 0$, the boundary conditions can be linearized to leading order and imposed on the horizontal line $y = 0$, $|x| > d(t)$. This is the main idea of the Wagner approach which has been successfully used both in theoretical analysis of water-impact problems and in practical applications for more than 70 years. Within the Wagner approach the function $d(t)$ is unknown and is determined as a part of the solution with the help of the so-called Wagner condition [1]. In the present problem the Wagner condition requires that the vertical position of the deformed free surface at $|x| = d(t)$ is zero. In the present analysis the Wagner solution is treated as the leading-order solution in the region I. The Wagner solution is briefly considered in Sect. 3.1.

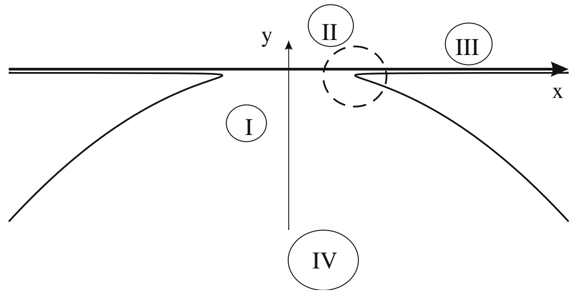

Fig. 3 The flow domain is divided into four regions I is the main flow region, II is the jet-root region, III is the jet region, IV is the far-field region. The asymptotic solutions are obtained for each region and are matched to one another

The flow in region IV, the dimension of which is of the order of $O(1)$ in non-dimensional variables, is uniform in the leading order as $t \to 0$. We need to determine higher-order terms of the solution in this region and match this solution to the higher-order solution in region I (see Sect. 3.2).

The flow in the jet-root region II, the dimension of which is much smaller than $d(t)$, was investigated for the first time by Wagner [1] to leading order as $d \to 0$. The corresponding solution was obtained in a form that is convenient for practical purposes in [2,3]. The matching of the inner solution in region II to the solution in the main flow region I was studied by Oliver [4]. The nonlinear boundary-value problem, which governs the flow in region II in the leading order, is revisited in Sect. 4 with the aim to estimate the order of accuracy of its solution during the initial stage.

The flow in the jet region III was studied in [2,3]. Here we use the results from [2] and [3], which show that the contribution of the pressure in the jet region to the total hydrodynamic force acting on the plate during the early stage of the impact is negligibly small compared with the contribution of the hydrodynamic pressure in the jet-root region II. Note that in [2] it was shown that the jet length is infinite within the incompressible-liquid model, which is why in the following we take $c(t) = \infty$.

It should be noted that we defined the scheme of the flow before the higher-order asymptotic solution has been obtained. This scheme is well suited for a water-impact analysis (see [3,5], for example). We take this scheme here for granted with the aim to improve the leading-order solution, which is well known [1], by taking into account the higher-order terms in the boundary conditions. Taking the higher-order terms in the water-impact problem into account is necessary due to the well-known discrepancy between the Wagner predictions of the hydrodynamic loads and the loads measured in experiments and computed within the fully nonlinear theory of potential flows induced by impact (see [6] for more details and discussions).

The second-order solution for the problem of wedge entry was studied by Fontaine and Cointe [7]. They derived the boundary-value problem for the second-order velocity potential in the region I and suggested to solve this problem numerically.

2 Formulation of the problem

The liquid flow after the impact instant, $t > 0$, is described in terms of non-dimensional variables by the velocity potential $\varphi(x, y, t)$, which satisfies the following equations

$$\Delta\varphi = 0 \quad (\text{in } \Omega(t)), \tag{1}$$

$$\varphi_y = 0 \quad (y = 0), \tag{2}$$

$$\varphi_y = Y_x(x,t)\varphi_x + Y_t(x,t), \quad p = 0, \quad (y = Y(x,t)), \tag{3}$$

$$p(x,y,t) = \frac{1}{2} - \frac{1}{2}|\nabla\varphi|^2 - \varphi_t, \quad (\text{in } \Omega(t)), \tag{4}$$

$$Y(x,t) \sim -\frac{1}{2}x^2 + t, \quad (|x| \to \infty), \tag{5}$$

$$\varphi \sim y \quad (x^2 + y^2 \to \infty), \tag{6}$$

$$Y(x,0) = -\frac{1}{2}x^2, \quad \varphi(x,y,0) = y. \tag{7}$$

Here the equation $y = Y(x,t)$ describes the deformed shape of the free surface after impact, $Y(x,t)$ is an unknown multi-valued function defined for $|x| > d(t)$ and $p(x,y,t)$ is the hydrodynamic pressure. The boundary condition (2) on the plate implies that the plate is fixed and rigid; the kinematic and dynamic boundary conditions (3) are imposed on the actual position of the free surface after impact. Conditions (5) and (6) represent the far-field conditions and imply that far from the impact region the free-surface deformations are negligibly small and the liquid flow is uniform. The unsteady Bernoulli equation (4) indicates that in the far field, where $x^2 + y^2 \gg 1$, the pressure is zero and the liquid moves vertically up as a rigid body. The unknown function $d(t)$ is defined by the condition

$$|Y_x(x,t)| \to \infty \quad (|x| \to d(t) + 0). \tag{8}$$

Note that we assume after Wilson [2] that the plate is totally wetted just after impact. This is a reasonable assumption once the liquid compressibility is not taken into account. Calculations of the jet length within the compressible liquid model can be found in [8].

It should be noted that there is no parameter in problem (1)–(8). The initial stage of the impact is defined as the stage, where the non-dimensional time is small, $t \ll 1$. In the dimensional variables this inequality implies that the displacements of the liquid particles are much smaller than the length scale R of the process during the initial stage under consideration.

We shall obtain the asymptotic solution of the boundary-value problem (1)–(8) as $t \to 0$. Note that $d(t) \to 0$ as $t \to 0$. It is convenient to take the quantity d as a small parameter of the problem and consider the time $t(d)$ as a new unknown function. Also, it is convenient to introduce the stretched variables

$$\xi = x/d, \quad \eta = y/d, \quad \phi(\xi,\eta,d) = \varphi/d, \quad q(\xi,\eta,d) = p/\dot{d},$$
$$H(\xi,d) = Y/d^2, \quad \chi(\xi,\eta,d) = \psi/d \tag{9}$$

and the stretched time

$$T(d) = t(d)/d^2, \tag{10}$$

where $\psi(x,y,t)$ is the stream function; $\psi(x,0,t) = \psi(0,y,t) = 0$ owing to the flow symmetry and the body boundary condition (2).

In the new stretched variables (9) and (10) the boundary-value problem (1)–(8) takes the form

$$\Delta\phi = 0, \quad (\text{in } \Omega_s(d)), \tag{11}$$

$$\phi_\eta = 0, \quad (\eta = 0), \tag{12}$$

$$\phi_\eta = \frac{1}{2S(d)}\left[2H - \xi H_\xi\right] + dH_\xi\phi_\xi + \frac{d}{2S(d)}H_d, \quad q = 0, \quad (\eta = dH(\xi,d)), \tag{13}$$

$$q(\xi,\eta,d) = \xi\phi_\xi + \eta\phi_\eta - \phi - d\phi_d + dS(d)\left[1 - |\nabla\phi|^2\right], \quad (\text{in } \Omega_s(d)), \tag{14}$$

$$H(\xi,d) \sim -\frac{1}{2}\xi^2 + T(d), \quad (|\xi| \to \infty), \tag{15}$$

$$\phi \sim \eta, \quad (\xi^2 + \eta^2 \to \infty), \tag{16}$$

$$|H_\xi| \to \infty, \quad (|\xi| \to 1 + 0), \tag{17}$$

where

$$S(d) = T(d) + \frac{1}{2}dT'(d), \quad d\dot{d} = \frac{1}{2S(d)}; \tag{18}$$

here $\Omega_s(d)$ is the flow region in the stretched variables. We shall determine the asymptotic solution of the problem (11)–(18) as $d \to 0$.

3 Asymptotic solution in the main flow region

Considering the flow and the pressure distribution in region I, one should keep in mind that non-physical behavior of the solution may be found close to the jet-root region II. The size of region II is much smaller than the size of region I. Therefore, region II should be treated as a point, when one deals with the flow in region I. The same is true for region IV. That is, the far-field behavior of the asymptotic solution in region I can be non-physical and, in this case, it must be corrected with the help of the asymptotic solution in region IV. Thus, a uniformly valid solution is obtained with the help of the method of matched asymptotic expansions.

The initial asymptotics of the solution in region I is sought in the form

$$\begin{aligned} \phi(\xi,\eta,d) &= \phi_0(\xi,\eta) + \Phi_1(\xi,\eta,d), \\ H(\xi,d) &= h_0(\xi) + H_1(\xi,d), \\ T(d) &= t_0 + T_1(d), \end{aligned} \tag{19}$$

where $\Phi_1(\xi,\eta,d), H_1(\xi,d)$ and $T_1(d)$ tend to zero together with their first derivatives as $d \to 0$. It should be noted that the asymptotic procedure is designed in such a way that the leading-order solution is identical to that given by Wagner [1]. Note that we do not specify the orders of the higher-order terms in the expansions (19) but shall determine these orders as part of the asymptotic solution.

3.1 Leading-order solution

By substituting (19) in Eqs. 11–18 and letting $d \to 0$, we obtain the equations which govern the leading-order solution

$$\Delta\phi_0 = 0, \quad (\eta < 0), \tag{20}$$

$$\frac{\partial \phi_0}{\partial \eta} = 0, \quad (\eta = 0,\ |\xi| < 1), \tag{21}$$

$$\frac{\partial \phi_0}{\partial \eta} = \frac{1}{2t_0}[2h_0 - \xi h_{0\xi}], \quad (\eta = 0,\ |\xi| > 1), \tag{22}$$

$$\xi\frac{\partial \phi_0}{\partial \xi} - \phi_0 = 0, \quad (\eta = 0,\ |\xi| > 1), \tag{23}$$

$$h_0(\xi) \sim -\frac{1}{2}\xi^2 + t_0, \quad (|\xi| \to \infty), \tag{24}$$

$$\phi_0 \sim \eta, \quad (\xi^2 + \eta^2 \to \infty), \tag{25}$$

$$|h_{0\xi}| \to \infty, \quad (|\xi| \to 1 + 0), \tag{26}$$

$$h_0(1) = 0. \tag{27}$$

Region III is of infinitesimal thickness with respect to the dimension of region I. This is why one can disregard the jets in the analysis of the flow in region I and impose the body boundary condition (12) only along the interval $|\xi| < 1$. Equation 27 follows from the adopted scheme of the flow and is known as the Wagner condition.

The general solution of the ordinary differential equation (23) is $\phi_0(\xi,0) = C_1\xi$, where $C_1 = 0$ according to condition (25). Therefore, the dynamic boundary condition (23) along the free surface provides

$$\phi_0 = 0 \quad (\eta = 0,\ |\xi| > 1). \tag{28}$$

The complex potential $\phi_0 + \mathrm{i}\chi_0$, which satisfies the boundary conditions (21) and (28) and the far-field condition (25) and has a minimal singularity at the "contact points" $\xi = \pm 1$, $\eta = 0$, being of the form

$$\phi_0 + \mathrm{i}\chi_0 = -\mathrm{i}\sqrt{z^2-1}, \quad z = \xi + \mathrm{i}\eta. \tag{29}$$

The complex function $\sqrt{z^2-1}$ is defined on the plane z with a cut along the interval $-1 < \xi < 1$, $\eta = 0$. The chosen branch of this function is defined as

$$\begin{aligned} \sqrt{z^2-1} &= \sqrt{\xi^2-1}, \qquad (\xi > 1, \;\; \eta = 0), \\ &= -\sqrt{\xi^2-1}, \quad (\xi < -1, \;\; \eta = 0), \\ &= -\mathrm{i}\sqrt{1-\xi^2}, \quad (|\xi| < 1, \;\; \eta = -0), \\ &= \mathrm{i}\sqrt{1-\xi^2}, \qquad (|\xi| < 1, \;\; \eta = +0). \end{aligned} \tag{30}$$

Along the free surface, $\xi > 1, \eta = 0$, we obtain $\phi_0 + \mathrm{i}\chi_0 = -\mathrm{i}\sqrt{\xi^2-1}$ and, therefore, $\chi_0(\xi, 0) = -\sqrt{\xi^2-1}$. By using the equality $\phi_{0\eta} = -\chi_{0\xi}$, we find $\phi_{0\eta} = \xi/\sqrt{\xi^2-1}$ along the free surface. The kinematic condition (22) and the conditions (26) and (27) lead to the following boundary-value problem for the function $h_0(\xi)$

$$\xi \frac{\partial h_0}{\partial \xi} - 2h_0 = -2t_0 \frac{\xi}{\sqrt{\xi^2-1}}, \;\; (\xi > 1), \tag{31}$$

$$h_0(1) = 0, \tag{32}$$

$$h_0(\xi) \sim -\frac{1}{2}\xi^2 + t_0, \quad (|\xi| \to \infty), \tag{33}$$

The general solution of Eq. 31 is

$$h_0(\xi) = C_2\xi^2 - 2t_0\xi\sqrt{\xi^2-1},$$

and condition (32) gives $C_2 = 0$. In the far field the obtained solution behaves as

$$h_0(\xi) \sim -2t_0\xi^2 + t_0 + O(\xi^{-2}),$$

which, together with condition (33), provides $t_0 = 1/4$ and the leading-order shape of the liquid free surface in the region I turns out to be

$$h_0(\xi) = -\frac{1}{2}\xi\sqrt{\xi^2-1}. \tag{34}$$

We obtained the leading-order solution in region I, which is identical to that obtained for the first time by Wagner [1].

3.2 Second-order solution

With the help of (19) the dynamic boundary condition, $q = 0$, on the free surface provides

$$\xi\left(\frac{\partial \phi_0}{\partial \xi}(\xi, dH) + \frac{\partial \Phi_1}{\partial \xi}(\xi, dH, d)\right) + dH(\xi, d)\left(\frac{\partial \phi_0}{\partial \eta}(\xi, dH) + \frac{\partial \Phi_1}{\partial \eta}(\xi, dH, d)\right)$$
$$-\phi_0(\xi, dH) - \Phi_1(\xi, dH, d) - d\frac{\partial \Phi_1}{\partial d}(\xi, dH, d) + dS(d)[1 - |\nabla\phi_0|^2(\xi, 0)] + O(d \cdot \Phi_1) = 0.$$

By using a Taylor expansion of terms in the latter equation and equality (28), we obtain in the leading order the dynamic condition as

$$d\xi\frac{\partial}{\partial \xi}\left(\frac{\partial \phi_0}{\partial \eta}\right)(\xi, 0)h_0(\xi) + \xi\frac{\partial \Phi_1}{\partial \xi}(\xi, 0, d) - \Phi_1(\xi, 0, d) - d\frac{\partial \Phi_1}{\partial d}(\xi, 0, d)$$
$$+\frac{d}{4}\left(1 - \left[\frac{\partial \phi_0}{\partial \eta}\right]^2(\xi, 0)\right) = o(d), \quad (|\xi| > 1). \tag{35}$$

Conditions (35), (12) and Eq. 11 show that

$$\Phi_1(\xi,\eta,d) = C_3(d)\phi_{1e}(\xi,\eta) + d\phi_1(\xi,\eta) + \Phi_2(\xi,\eta,d), \tag{36}$$

where $\Phi_2(\xi,\eta,d) = o(d)$ and $\Phi_2(\xi,\eta,d) = o(C_3(d))$ as $d \to 0$; the function $\phi_{1e}(\xi,\eta)$ is a non-trivial solution of the boundary-value problem

$$\Delta\phi_{1e} = 0 \quad (\eta < 0), \quad \phi_{1e} \to 0, \quad (\xi^2+\eta^2 \to \infty),$$
$$\frac{\partial\phi_{1e}}{\partial\eta} = 0 \quad (\eta = 0,\ |\xi| < 1), \quad \phi_{1e} = 0, \quad (\eta = 0,\ |\xi| > 1).$$

Possible solutions of this problem are singular at the contact points. The solution of minimal singularity has the form

$$\phi_{1e}(\xi,\eta) = \mathfrak{Re}[1/\sqrt{1-z^2}].$$

The function $\phi_{1e}(\xi,\eta)$ could be required to match the second-order outer solution with the inner solution in the jet-root region. However, the following analysis shows that the matching procedure does not allow singular terms in the second-order solution for the velocity potential, which is why we take $C_3(d) = o(d)$ as $d \to 0$.

The potential $\phi_1(\xi,\eta)$ satisfies the body boundary condition (12) and the condition on the free surface

$$\xi\frac{\partial\phi_1}{\partial\xi} - 2\phi_1 = \frac{1}{4}\left[\phi_{0\eta}^2 - 1\right] - \xi h_0(\xi)[\phi_{0\eta}]_\xi, \quad (\eta = 0,\ \ |\xi| > 1), \tag{37}$$

which follows from (35) and (36). Condition (37) leads to the equation

$$\xi\frac{\partial\phi_1}{\partial\xi} - 2\phi_1 = -\frac{1}{2} - \frac{1}{4}\frac{1}{\xi^2-1}, \quad (\eta = 0,\ \ \xi > 1)$$

with its general solution

$$\phi_1(\xi,0) = C_4\xi^2 + \frac{1}{8}\left[1 - \xi^2\log\left(1 - \frac{1}{\xi^2}\right)\right], \quad (\xi > 1). \tag{38}$$

Note that

$$\phi_1(\xi,0) = C_4\xi^2 + \frac{1}{4} + \frac{1}{16}\xi^{-2} + O(\xi^{-4}) \tag{39}$$

as $\xi \to +\infty$ and

$$\phi_1(\xi,0) = -\frac{1}{8}\log(\xi-1) + O(1) \tag{40}$$

as $\xi \to 1+0$. The validity of the solution can be checked by substitution in (37).

In the original variables the term with C_4 in (38) gives the contribution to the velocity potential on the free surface as $d \cdot d \cdot C_4(x/d)^2 = C_4x^2$, which does not decay for $t \to 0$. This result in combination with the initial conditions (7) gives $C_4 = 0$. The same result can be also obtained by matching the potential distribution (38) along the free surface to that in region IV (see [9], where the far-field asymptotics of the solution has been obtained for the problem of liquid-drop impact in the leading order as $t \to 0$). Without giving details, we present here only the final result for the velocity potential in region IV written with the help of the stretched variables

$$\phi(\xi,\eta,d) = \eta + \frac{1}{2}d\left[\frac{\eta}{\xi^2+\eta^2} + \frac{1}{2}\right] + O(d^3), \quad (d \to 0).$$

It is clear that the latter asymptotics matches the asymptotic expansion (39) if and only if $C_4 = 0$.

The kinematic boundary condition (13) and Eqs. 19, 22 and 36 provide

$$H(\xi,d) = h_0(\xi) + dh_1(\xi) + H_2(\xi,d), \tag{41}$$

$$T(d) = \frac{1}{4} + dt_1 + T_2(d), \tag{42}$$

where $d^{-1}H_2(\xi,d) \to 0$ and $d^{-1}T_2(d) \to 0$ as $d \to 0$. A Taylor expansion of the terms in the kinematic condition (13) as $d \to 0$ with (22) taken into account gives

$$\phi_{1\eta}(\xi,0) = 2(3h_1 - \xi h_1') - 6t_1\phi_{0\eta}, \quad (\eta = 0, |\xi| > 1),$$

which is considered in the following as an ordinary differential equation with respect to the second-order elevation of the free surface $h_1(\xi)$:

$$\xi\frac{\partial h_1}{\partial \xi} - 3h_1 = -3t_1\phi_{0\eta} - \frac{1}{2}\phi_{1\eta}(\xi,0), \quad (\xi > 1). \tag{43}$$

In this equation the coefficient t_1 and the normal derivative $\phi_{1\eta}(\xi,0)$ along the free surface are still unknown.

The derivative $\phi_{1\eta}(\xi,0)$ in (43) is obtained as a part of the solution of the boundary-value problem for the second-order velocity potential $\phi_1(\xi,\eta)$

$$\Delta\phi_1 = 0, \quad (\eta < 0), \tag{44}$$

$$\frac{\partial \phi_1}{\partial \eta} = 0, \quad (\eta = 0, \ |\xi| < 1), \tag{45}$$

$$\phi_1 = \frac{1}{8} - \frac{1}{8}\xi^2 \log\left(1 - \frac{1}{\xi^2}\right), \quad (\eta = 0, \ |\xi| > 1), \tag{46}$$

$$\phi_1 \to \frac{1}{4}, \quad (\xi^2 + \eta^2 \to \infty), \tag{47}$$

Details of the analysis of the boundary-value problem (44)–(47) can be found in the Appendix. Here we present only the final results

$$\frac{\partial \phi_1}{\partial \eta}(\xi,0) = \frac{1}{2}\left\{|\xi| \arctan\left[\frac{1}{\sqrt{\xi^2-1}}\right] - \frac{|\xi|}{\sqrt{\xi^2-1}}\right\}, \quad (|\xi| > 1), \tag{48}$$

$$\phi_1(\xi,0) = \frac{1}{8} + \frac{1}{4}\sqrt{1-\xi^2} - \frac{1}{8}\xi^2\log(1-\xi^2) + \frac{1}{4}\xi^2\log\left[1 + \sqrt{1-\xi^2}\right], \quad (|\xi| < 1). \tag{49}$$

Now the right-hand side of Eq. 43 is defined and we can integrate this equation.

3.3 Second-order free-surface elevation

We denote the right-hand side of Eq. 43 as $G(\xi,t_1)$, where

$$G(\xi,t_1) = -3t_1\frac{\xi}{\sqrt{\xi^2-1}} - \frac{\xi}{4}\left\{\arctan\left[\frac{1}{\sqrt{\xi^2-1}}\right] - \frac{1}{\sqrt{\xi^2-1}}\right\}.$$

The general solution of the ordinary differential equation (43) has the form

$$h_1(\xi) = C_5\xi^3 - \xi^3\int_\xi^\infty G(\alpha,t_1)\frac{\mathrm{d}\alpha}{\alpha^4}.$$

The condition at the infinity, $h_1(\xi) \to 0$ as $\xi \to \infty$, gives $C_5 = 0$ and

$$h_1(1) = -\int_1^\infty G(\alpha,t_1)\frac{\mathrm{d}\alpha}{\alpha^4}.$$

By using the integrals

$$\int_1^\infty \left\{\arctan\left[\frac{1}{\sqrt{\xi^2-1}}\right] - \frac{1}{\sqrt{\xi^2-1}}\right\}\frac{\mathrm{d}\xi}{\xi^3} = -\frac{\pi}{8}, \quad \int_1^\infty \frac{\mathrm{d}\xi}{\xi^3\sqrt{\xi^2-1}} = \frac{\pi}{4},$$

we obtain

$$h_1(1) = -\frac{3\pi}{4}t_1 - \frac{\pi}{32}. \tag{50}$$

The constant t_1, which defines the second-order wetting correction by Eq. 42, is evaluated by matching the second-order free-surface elevation in the main flow region I with the free-surface shape in the jet-root region II (see Sect. 4).

3.4 Second-order hydrodynamic pressure

The pressure distribution in the main part of the contact region, $|\xi| < 1$, is given by Eq. 14, which, by taking account of (12), provides

$$q(\xi,0,d) = \xi\phi_\xi(\xi,0,d) - \phi(\xi,0,d) - d\phi_d(\xi,0,d) + dS(d)[1 - \phi_\xi^2(\xi,0,d)], \tag{51}$$

where $dS(d) = \frac{1}{4}d + O(d^2)$. In the main flow region I,

$$\phi(\xi,0,d) = \phi_0(\xi,0) + d\phi_1(\xi,0) + O(d^2).$$

Therefore,

$$q(\xi,0,d) = q_0(\xi) + dq_1(\xi) + O(d^2). \tag{52}$$

The leading-order term of the pressure $q_0(\xi)$ is given as

$$q_0(\xi) = \xi\phi_{0\xi}(\xi,0) - \phi_0(\xi,0) = \frac{1}{\sqrt{1-\xi^2}},$$

which is the well-known pressure distribution obtained by the Wagner theory [1]. We used the leading-order velocity-potential distribution in the contact region, $\phi_0(\xi,0) = -\sqrt{1-\xi^2}$, which follows from (29). The second-order pressure distribution $q_1(\xi)$, in the main part of the contact region is calculated with the help of (49) as (see Appendix for details)

$$q_1(\xi) = \xi\phi_{1\xi}(\xi,0) - 2\phi_1(\xi,0) + \frac{1}{4}\left[1 - \phi_{0\xi}^2\right] = -\frac{1}{2\sqrt{1-\xi^2}}.$$

Finally, we obtain

$$q(\xi,0,d) = \frac{1-d/2}{\sqrt{1-\xi^2}} + O(d^2). \tag{53}$$

Despite the complex form of the second-order velocity-potential distribution (49), the second-order pressure distribution in the main part of the contact region differs only by the coefficient from the pressure distribution given by the Wagner theory.

3.5 Asymptotic behavior of the second-order velocity potential close to the contact points

The second-order velocity potential is singular at the contact points $\xi = \pm 1$. In order to resolve the singularity and describe the fine details of the flow and the pressure distribution in the jet-root region II, one needs to derive the inner asymptotic solution in region II and match it to the second-order solution in the main flow region I. In the following the latter solution is referred to as the outer solution. In this section we obtain the asymptotic behavior of the outer solution close to the contact point $\xi = 1$.

Along the rigid plate, $\eta = 0$, the second-order velocity potential as $\xi = 1 - \varrho a(d)$, where $\varrho > 0$ and $a(d) \to 0$ as $d \to 0$, behaves as

$$\begin{aligned}\phi(\xi, 0, d) &= -\sqrt{1-\xi^2} + d\left\{\frac{1}{8} + \frac{1}{4}\sqrt{1-\xi^2} - \frac{1}{8}\xi^2 \log\left(1-\xi^2\right) + \frac{1}{4}\xi^2 \log\left[1+\sqrt{1-\xi^2}\right]\right\} + O(d^2) \\ &= -\sqrt{2-\varrho a}\sqrt{\varrho a} + d\left\{\frac{1}{8} + \frac{1}{4}\sqrt{2-\varrho a}\sqrt{\varrho a} - \frac{1}{8}\left(1 - 2\varrho a + \varrho^2 a^2\right)[\log(2-\varrho a) + \log(\varrho a)]\right. \\ &\quad \left. + \frac{1}{4}\left(1 - 2\varrho a + \varrho^2 a^2\right)\log\left[1 + \sqrt{2-\varrho a}\sqrt{\varrho a}\right]\right\} + O(d^2). \end{aligned} \tag{54}$$

The first and second-order terms in (54) are of the same order as $d \to 0$, if and only if $a(d) = d^2$. Note that in comparing the term orders we do not account for the terms that are independent of the spatial coordinate ϱ. Substituting $a(d) = d^2$ in (54), we obtain

$$\phi(\xi, 0, d) = d\left\{-\sqrt{2\varrho} - \frac{1}{8}\log\varrho + K(d)\right\} + O(d^2), \quad K(d) = \frac{1}{8}(1 - \log 2) - \frac{1}{4}\log d. \tag{55}$$

When we approach the contact point along the free-surface, $\xi = 1 + \varrho d^2$, $\varrho > 0$, the asymptotic behavior of the velocity potential is

$$\begin{aligned}\phi(\xi, dH(\xi, d), d) &= \phi_0(\xi, dH) + d\phi_1(\xi, 0) + O(d^2) \\ &= \phi_0(\xi, 0) + \phi_{0\eta}(\xi, 0)\cdot dh_0(\xi) + d\phi_1(\xi, 0) + O(d^2) \\ &= d\left\{-\frac{1}{2} - \frac{1}{8}\log\varrho + K(d)\right\} + O(d^2). \end{aligned} \tag{56}$$

Asymptotic formulae (55) and (56) are used below in a matching procedure to derive the leading-order inner solution, which describes details of the flow close to the contact points where the outer asymptotic solution is not valid.

4 Asymptotic solution in the jet-root region

The leading-order inner solution was derived several times starting from the original paper by Wagner [1]. In the present analysis we pay special attention to the accuracy of the inner solution as $d \to 0$. The inner variables are introduced as

$$\lambda = \frac{\xi - 1}{d^2}, \quad \mu = \frac{\eta}{d^2}, \quad \phi = d\Phi_i(\lambda, \mu, d), \quad H = d\zeta(\lambda, d), \quad q = \frac{1}{d}Q(\lambda, \mu, d). \tag{57}$$

Within the inner variables (57) the boundary-value problem (11)–(14), together with the matching conditions (55) and (56), where now $\varrho = \sqrt{\lambda^2 + \mu^2}$ and $\varrho \gg 1$, $d \ll 1$, takes the form

$$\Delta\Phi_i = 0, \quad (\text{in } \Omega_{\text{inner}}(d)), \tag{58}$$

$$\Phi_{i\mu} = 0 \quad (\mu = 0), \tag{59}$$

$$\Phi_{i\mu} = \zeta_\lambda \Phi_{i\lambda} - \frac{1}{2S(d)}\zeta_\lambda + \frac{d^2}{2S(d)}D_\zeta, \quad (\mu = \zeta(\lambda, d)), \tag{60}$$

$$Q(\lambda, \mu, d) = \Phi_{i\lambda} - S(d)|\nabla\Phi_i|^2 + d^2 D_q, \quad (\text{in } \Omega_{\text{inner}}(d)), \tag{61}$$

$$Q = 0, \quad (\mu = \zeta(\lambda, d)), \tag{62}$$

$$\Phi_i \sim -\sqrt{2|\lambda|} - \frac{1}{8}\log|\lambda| + K(d), \quad (\mu = 0,\ \lambda \to -\infty), \tag{63}$$

$$\Phi_i \sim -\frac{1}{2} - \frac{1}{8}\log\lambda + K(d), \quad (\mu = \zeta(\lambda, d),\ \lambda \to \infty), \tag{64}$$

where

$D_\zeta = 3\zeta - 3\lambda\zeta_\lambda + d\zeta_d, \quad D_q = 3\lambda\Phi_{i\lambda} + 3\mu\Phi_{i\mu} - 2\Phi_i + S(d) - d\Phi_{id}.$

It is convenient to introduce the modified velocity potential $\tilde{\Phi}(\lambda, \mu, d)$ as

$$\Phi_i(\lambda, \mu, d) = \frac{1}{2S(d)}[\lambda + \tilde{\Phi}(\lambda, \mu, d)]. \tag{65}$$

The boundary-value problem (58)–(64) written with respect to the new unknown functions takes the form

$$\Delta\tilde{\Phi} = 0, \quad (\text{in } \Omega_{\text{inner}}(d)), \tag{66}$$

$$\tilde{\Phi}_\mu = 0, \quad (\mu = 0), \tag{67}$$

$$\tilde{\Phi}_\mu = \zeta_\lambda\tilde{\Phi}_\lambda + d^2 D_\zeta, \quad (\mu = \zeta(\lambda, d)), \tag{68}$$

$$Q(\lambda, \mu, d) = \frac{1}{4S(d)}\left[1 - |\nabla\tilde{\Phi}|^2\right] + d^2\tilde{D}_q, \quad (\text{in } \Omega_{\text{inner}}(d)), \tag{69}$$

$$|\nabla\tilde{\Phi}|^2 = 1 + 4d^2 S(d)\tilde{D}_q, \quad (\mu = \zeta(\lambda, d)), \tag{70}$$

$$\tilde{\Phi} \sim -2S\sqrt{2|\lambda|} - \lambda - \frac{S(d)}{4}\log|\lambda| + 2S(d)K(d) + O(d), \quad (\mu = 0, \lambda \to -\infty), \tag{71}$$

$$\tilde{\Phi} \sim -\lambda - S(d) - \frac{S(d)}{4}\log\lambda + 2S(d)K(d) + O(d), \quad (\mu = \zeta(\lambda, d), \lambda \to \infty), \tag{72}$$

where the terms $O(d)$ in (71) and (72) come from the third-order outer solution,

$$\tilde{D}_q = \frac{3\lambda}{2S(d)}[1 + \tilde{\Phi}_\lambda] + \frac{3\mu}{2S(d)}\tilde{\Phi}_\mu - \frac{1}{S(d)}[\lambda + \tilde{\Phi}] + S(d) - \frac{d}{2S}[\tilde{\Phi}_d - S'(d)(\lambda + \tilde{\Phi})/S].$$

Omitting in the inner problem (66)–(72) the terms of the order of $O(d^2)$, we arrive at the approximate boundary-value problem for the modified velocity potential

$$\begin{aligned} &\Delta\tilde{\Phi} = 0, \quad (\text{in } \Omega_{\text{inner}}(d)), \\ &\tilde{\Phi}_\mu = 0, \quad (\mu = 0), \\ &|\nabla\tilde{\Phi}|^2 = 1, \quad \tilde{\Phi}_n = 0, \quad (\mu = \zeta(\lambda, d)) \end{aligned} \tag{73}$$

with the matching conditions (71) and (72), where $\tilde{\Phi}_n$ is the normal derivative of the modified potential along the free-surface. The condition $\tilde{\Phi}_n = 0$ along the free-surface implies that the free-surface is approximately a streamline in the jet-root region II with accuracy up to $O(d^2)$ as $d \to 0$. Equations 73 can be used also in combination with the third-order outer solution, when the matching conditions include the terms of the order of $O(d)$. In this paper, we limit ourselves to the second-order outer solution.

The inner problem (73) was solved several times (see [1–4], for example). Here we briefly reproduce the solution procedure with the aim to demonstrate the matching of the leading-order inner solution with the second-order outer solution. In this analysis d is treated as a parameter. By using the complex potential $\tilde{W}(\omega) = \tilde{\Phi} + \mathrm{i}\tilde{\Psi}$, where $\omega = \lambda + \mathrm{i}\mu$ and $\tilde{\Psi}(\lambda, \mu)$ is the stream function, and the complex velocity $U(\omega) = \tilde{W}'(\omega) = \tilde{\Phi}_\lambda - \mathrm{i}\tilde{\Phi}_\mu$, the problem (73) can be reformulated on the hodograph plane U as: *To find the analytic function $\tilde{W}(U)$ in the region bounded by the line $\Im\mathfrak{m}U = 0$ and the curve $|U| = 1$, which satisfies the boundary conditions $\Im\mathfrak{m}\tilde{W} = 0$ on the body surface, where $\Im\mathfrak{m}U = 0$, and $\Im\mathfrak{m}\tilde{W} = \tilde{\Psi}_{fs}$ along the free-surface, where $|U| = 1$, and the far-field condition as $U \to -1$, which follows from the matching conditions* (71) *and* (72). Here the constant value $\tilde{\Psi}_{fs}$ of the stream function along the free-surface is unknown in advance and has to be determined as a part of the solution. On the hodograph plane a small vicinity of the corner point $U = 1$ corresponds to the origin of the jet region III, where the inner solution should be matched with the solution in the jet region III.

In order to derive the far-field condition at the corner point $U = -1$, we take in the matching conditions (71), (72) only the leading-order terms in the far field

$$\tilde{W} \sim -2^{\frac{3}{2}} S(d)\mathrm{i}\sqrt{\omega} - \omega \quad (|\omega| \to \infty). \tag{74}$$

Here $\sqrt{\omega} = \sqrt{\lambda}$, where $\mu = 0$, $\lambda > 0$, and $\sqrt{\omega} = -\mathrm{i}\sqrt{|\lambda|}$, where $\mu = -0$, $\lambda < 0$. Differentiating (74) with respect to ω, we find

$$U \sim -1 - \sqrt{2}S(d)\mathrm{i}/\sqrt{\omega}$$

and

$$(U+1)^2 \sim -\frac{2S^2}{\omega} \sim \frac{2S^2}{\tilde{W}}$$

in the far field. The latter equation can be presented as

$$\tilde{W} \sim \frac{2S^2(d)}{(U+1)^2} \quad (U \to -1), \tag{75}$$

which is the required far-field condition at the corner point $U = -1$.

Once the analytic function $\tilde{W}(U)$, which satisfies condition (75) and the corresponding boundary conditions, has been obtained, one can derive the solution of the original problem in parametric form if the function $\omega(U)$ is known. This function is computed by integrating the equation

$$\mathrm{d}\omega = \frac{\mathrm{d}\tilde{W}}{U} = \frac{\mathrm{d}\tilde{W}}{\mathrm{d}U}\frac{\mathrm{d}U}{U} = \tilde{W}'(U)\frac{\mathrm{d}U}{U}.$$

Note that the point $U = 0$ on the hodograph plane corresponds to the stagnation point in the inner coordinate system. The stagnation point is in the jet-root region but not in the far field if and only if $\tilde{W}'(0) = 0$. The latter equation is used in the following analysis to find the constant value of the stream function $\tilde{\Psi}_{fs}$ along the free-surface, $|U| = 1$.

One can check that the analytic function

$$\tilde{W}(U) = -2S^2\left\{\frac{U}{(U+1)^2} + \frac{1}{2}\log\left(\frac{1-U}{1+U}\right)\right\} + L(d) \tag{76}$$

satisfies the far-field condition (75) and the equation $\tilde{W}'(0) = 0$, its imaginary part being constant along the free-surface, $|U| = 1$, and zero on the body surface, $\mathfrak{Im}U = 0$. Here $L(d)$ is a real function, which has to be determined with the help of the matching conditions (71) and (72).

The function $\omega(U)$ is obtained by integration of the equation

$$\frac{1}{2S^2}\frac{\mathrm{d}\omega}{\mathrm{d}U} = \frac{\tilde{W}'(U)}{2S^2U} = -\frac{1}{U(1+U)^2} + \frac{2}{(1+U)^3} + \frac{1}{2U(1-U)} + \frac{1}{2U(1+U)}.$$

We find

$$\frac{\omega(U)}{2S^2} = -\frac{1}{1+U} - \frac{1}{(1+U)^2} - \frac{1}{2}\log\left(\frac{1-U}{1+U}\right) + C_6. \tag{77}$$

Along the free-surface, where $U = \exp(-\mathrm{i}\theta)$, $0 < \theta < \pi$, we obtain $(C_6 = \mathfrak{Re}(C_6) + \mathrm{i}\mathfrak{Im}(C_6))$

$$\frac{\lambda + \mathrm{i}\mu}{2S^2} = -\frac{3}{4} + \frac{1}{4}\tan^2\frac{\theta}{2} - \frac{1}{2}\log\left(\tan\frac{\theta}{2}\right) + \mathfrak{Re}(C_6) + \mathrm{i}\left[-\tan\frac{\theta}{2} - \frac{\pi}{4} + \mathfrak{Im}(C_6)\right]. \tag{78}$$

At the point P of the free-surface, where the tangential to the free-surface is vertical and the horizontal component of the flow velocity in the inner coordinate system is zero, $\mathfrak{Re}U = 0$ and $\theta = \pi/2$, Eq. 78 gives

$$\frac{\lambda + \mathrm{i}\mu}{2S^2} = -\frac{1}{2} + \mathfrak{Re}(C_6) + \mathrm{i}\left[-1 - \frac{\pi}{4} + \mathfrak{Im}(C_6)\right].$$

The inner coordinate system has been defined in such a way that $\lambda = 0$ at the point P of the free-surface, which yields

$$\mathfrak{Re}(C_6) = \frac{1}{2}.$$

Along the rigid surface $U = \alpha$, $-1 < \alpha < 1$ and $\mu = 0$. The imaginary part of the (77) provides

$\mathfrak{Im}(C_6) = 0.$

Therefore, along the free-surface

$$\frac{\lambda + \mathrm{i}\mu}{2S^2} = -\frac{1}{4} + \frac{1}{4}\tan^2\frac{\theta}{2} - \frac{1}{2}\log\left(\tan\frac{\theta}{2}\right) + \mathrm{i}\left[-\tan\frac{\theta}{2} - \frac{\pi}{4}\right], \tag{79}$$

and along the rigid surface

$$\frac{\lambda}{2S^2} = -\frac{1}{1+\alpha} - \frac{1}{(1+\alpha)^2} - \frac{1}{2}\log\left(\frac{1-\alpha}{1+\alpha}\right) + \frac{1}{2}, \quad \tilde{\Phi}_\lambda = \alpha. \tag{80}$$

Equation 80, in particular, gives the coordinate λ_s of the stagnation point, where $\tilde{\Phi}_\lambda(\lambda_s, 0) = 0$ as

$$\lambda_s = -3S^2. \tag{81}$$

In the limit as $\theta \to 0$ Eq. 79 provides the jet thickness h_j in the inner variables as

$$\frac{\mu(1)}{2S^2} = -\frac{h_j}{2S^2} = -\frac{\pi}{4}.$$

Therefore,

$$h_j = \frac{\pi}{2}S^2(d).$$

We obtain the vertical coordinate μ_P of the point P on the free-surface as

$$\mu_P = -h_j - 2S^2. \tag{82}$$

Along the free-surface Eq. 76 gives

$$\tilde{\Phi} = -2S^2\left\{\frac{1}{4\cos^2(\theta/2)} + \frac{1}{2}\log\left(\tan\frac{\theta}{2}\right)\right\} + L(d), \quad \tilde{\Psi} = -h_j \tag{83}$$

and along the rigid surface

$$\tilde{\Phi} = -2S^2\left\{\frac{\alpha}{(1+\alpha)^2} + \frac{1}{2}\log\left(\frac{1-\alpha}{1+\alpha}\right)\right\} + L(d), \quad \tilde{\Psi} = 0, \quad \tilde{\Phi}_\lambda = \alpha. \tag{84}$$

In order to determine the function $L(d)$ in (83), (84) and to demonstrate matching of the velocity potential distributions (83) and (84) with the velocity potential in the outer region I, we need to obtain the asymptotic behavior of the velocity potential along both the free-surface and the rigid surface in the far field.

In the far field along the rigid surface, where $\lambda \to -\infty$ and $\alpha \to -1 + 0$, it is convenient to introduce a small positive parameter $\varepsilon = 1 + \alpha$ and determine the asymptotic behavior of the functions (80) and (84) as $\varepsilon \to 0$. Equations 80 and 84 provide as $\lambda \to -\infty$

$$\frac{1}{\varepsilon} = \frac{1}{S}\sqrt{\frac{|\lambda|}{2}} - \frac{1}{2} + O\left(\frac{\log|\lambda|}{\sqrt{|\lambda|}}\right),$$

$$\tilde{\Phi} = -2S^2\left\{\frac{\lambda}{2S^2} + \frac{2}{\varepsilon} + \log(2-\varepsilon) - \log\varepsilon - \frac{1}{2}\right\} + L(d).$$

By distinguishing in the latter equations the higher-order terms (h.o.t.), which vanish as $\lambda \to -\infty$ and $d \to 0$, and using the asymptotic formula $S(d) = \frac{1}{4} + O(d)$, we obtain in the far field of the inner region

$$\tilde{\Phi} \sim -2S\sqrt{2|\lambda|} - \lambda - \frac{S(d)}{4}\log|\lambda| + \frac{3}{16} - \frac{5}{16}\log 2 + L(d) + h.o.t. \quad (\mu = 0,\ \lambda \to -\infty) \tag{85}$$

Comparing (71) and (85), we obtain

$$L(d) = \frac{1}{4}\log 2 - \frac{1}{8}[1 + \log d]. \tag{86}$$

Equations 79 and 83 give the modified potential distribution along the free-surface as

$$\tilde{\Phi} = -\lambda - S^2 - 2S^2 \log\left(\tan\frac{\theta}{2}\right) + L(d),$$

where

$$\tan\frac{\theta}{2} = \frac{\sqrt{2\lambda}}{S} + O\left(\frac{\log\lambda}{\sqrt{\lambda}}\right)$$

as $\lambda \to +\infty$. These equations provide in the limits $\lambda \to +\infty$ and $d \to 0$ with account for (86)

$$\tilde{\Phi} \sim -\lambda - S(d) - \frac{S(d)}{4}\log\lambda + \frac{1}{16} - \frac{1}{8}\log d - \frac{1}{16}\log 2 + \text{h.o.t.} \quad (\mu = \zeta(\lambda, d),\ \lambda \to \infty),$$

which coincides with the asymptotic behavior of the outer potential along the free-surface given by Eq. 72.

In the far-field equation (79) provides the asymptotic behavior of the inner free-surface shape as

$$\mu = -\frac{1}{2}\sqrt{2\lambda} - \frac{\pi}{32} + \text{h.o.t.} \quad (\lambda \to \infty). \tag{87}$$

This asymptotic formula should be matched with the second-order outer solution for the free-surface shape. With the help of (34), (41) and (57) we can present the outer solution as

$$\eta = dH(\xi, d) = d[h_0(\xi) + dh_1(\xi) + O(d^2)].$$

In the inner variables this equation gives

$$d^2\mu = d\left[h_0\left(1 + d^2\lambda\right) + dh_1\left(1 + d^2\lambda\right) + O\left(d^2\right)\right]$$

and in the leading order as $d \to 0$

$$\mu = -\frac{1}{2d}(1 + d^2\lambda)\sqrt{d^2\lambda(2 + d^2\lambda)} + h_1(1 + d^2\lambda) + O(d) = -\frac{1}{2}\sqrt{2\lambda} + h_1(1) + O(d). \tag{88}$$

The outer asymptotic shape of the free-surface (88) matches the far-field asymptotics of the inner free-surface shape (87) if

$$h_1(1) = -\frac{\pi}{32}$$

and formula (50) provides

$$t_1 = 0. \tag{89}$$

Equations 10 and 42 lead to the asymptotic formula

$$d(t) = 2\sqrt{t} + O\left(t^{\frac{3}{2}}\right) \quad (t \to 0), \tag{90}$$

which explains why the dimension of the contact region predicted by the Wagner theory, $d(t) = 2\sqrt{t}$ in the problem under consideration, is in good agreement with experimental data. We obtained that, in the problem of wave impact, the second-order solution does not change the dimension of the wetted area given by the first-order Wagner approach.

5 Second-order hydrodynamic force

The non-dimensional hydrodynamic force acting onto the rigid plate $F(t)$, where $\rho V^2 R$ is the force scale, is given with the help of (9) and (18) as

$$F(t) = \int_{-\infty}^{\infty} p(x, 0, t)\mathrm{d}x = \frac{1}{S(d)}\int_0^{\infty} q(\xi, 0, d)\mathrm{d}\xi. \tag{91}$$

During the initial impact stage, when $d \ll 1$, the pressure distributions in both the main part of the contact region, $|\xi| < 1$, and in the jet-root region II have to be taken into account in the integral (91). In order to do this, we introduce a large parameter L, where $L \gg 1$, $d^2 L \ll 1$, and present Eq. 91 in the form

$$F(t) = \frac{1}{S(d)} \left\{ \int_0^{1-d^2L} q(\xi, 0, d) \mathrm{d}\xi + \int_{1-d^2L}^{\infty} q(\xi, 0, d) \mathrm{d}\xi \right\}. \tag{92}$$

The first integral in (92) is calculated by using the asymptotic formula (53) and the second integral is calculated by using Eqs. 57, 67, 69 and 84.

The first integral in (92) provides the contribution $F_{\text{outer}}(t)$ of the second-order outer solution to the hydrodynamic force

$$\begin{aligned} F_{\text{outer}}(t) &= \frac{1}{S(d)} \left(1 - \frac{d}{2}\right) \int_0^{1-d^2L} \frac{\mathrm{d}\xi}{\sqrt{1-\xi^2}} + O(d^2) \\ &= (4 + O(d^2)) \left(1 - \frac{d}{2}\right) \left[\frac{\pi}{2} - \sqrt{2d^2L} + O\left([d^2L]^{\frac{3}{2}}\right)\right] + O(d^2) \\ &= 2\pi - 4d\sqrt{2L} - \pi d + O\left(d^2\sqrt{L}\right). \end{aligned} \tag{93}$$

The first term in (93) is the hydrodynamic force given by the first-order Wagner theory [1].

The contribution $F_{\text{inner}}(t)$ of the jet-root region II to the second-order total hydrodynamic force is evaluated as

$$F_{\text{inner}}(t) = \frac{1}{S(d)} \int_{1-d^2L}^{\infty} q(\xi, 0, d) \mathrm{d}\xi = \frac{d}{S(d)} \int_{-L}^{\infty} Q(\lambda, 0, d) \mathrm{d}\lambda.$$

Equations 67 and 69 give

$$Q(\lambda, 0, d) = \frac{1}{4S(d)} [1 - \tilde{\Phi}_\lambda^2(\lambda, 0, d)] + d^2 \tilde{D}_q,$$

where $\tilde{\Phi}(\lambda, 0, d)$ is given by Eq. 84. It is convenient to change the integration variable for α with the help of (80). These equations yield

$$\frac{\mathrm{d}\lambda}{2S^2} = \frac{4\mathrm{d}\alpha}{(1-\alpha^2)(1+\alpha)^2}, \quad \tilde{\Phi}_\lambda(\lambda, 0, d) = \alpha + O(d).$$

We obtain

$$F_{\text{inner}}(t) = 2d \int_{\alpha(-L)}^{1} \frac{\mathrm{d}\alpha}{(1+\alpha)^2} + O(d^2), \tag{94}$$

where $\alpha(-L)$ is the solution of Eq. 80 for $\lambda = -L$. By using the asymptotic formula for $1/(1+\alpha)$, which was derived in Sect. 4, we find

$$\frac{1}{1+\alpha(-L)} = \frac{1}{S}\sqrt{\frac{L}{2}} - \frac{1}{2} + O\left(\frac{\log L}{\sqrt{L}}\right) \quad (L \to \infty). \tag{95}$$

Evaluating the integral in (94) and substituting in the result the asymptotic formula (95), we find

$$F_{\text{inner}}(t) = -2d + 4d\sqrt{2L} + O\left(d \log L/\sqrt{L}\right). \tag{96}$$

By neglecting the higher-order terms, the total hydrodynamic force is obtained as

$$F(t) = 2\pi - 2(\pi + 2)\sqrt{t} + \text{h.o.t.} \tag{97}$$

Comparing (93) and (96), one can conclude that the contribution of the jet-root region II into the total hydrodynamic force is of second order, that is, $F_{\text{inner}}(t)$ is of the same order as the second-order term $-\pi d$ in the asymptotic formula (93) for $F_{\text{outer}}(t)$. Therefore, account for details of the pressure distribution close

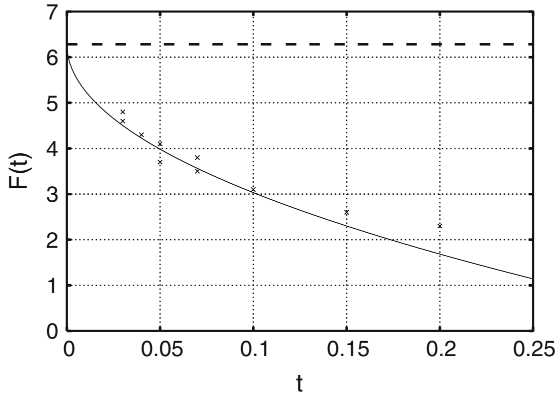

Fig. 4 The non-dimensional hydrodynamic force for the problem of circular cylinder entry at constant velocity: solid line is given by asymptotic formula (97), points represent the experimental results by Cointe and Armand [11], the classical Wagner theory predicts $F(t) = 2\pi$, which is shown by the dashed line

to the periphery of the contact region in combination with the first-order outer solution may provide the wrong estimate of the hydrodynamic force and the second-order outer force component is required.

We cannot prove rigorously at this moment that the obtained formula for the hydrodynamic force (97) can be applied also to the problem of a rigid parabolic contour entering water at constant velocity. Within the first-order Wagner theory it is well-known (see [10], for example) that the relative vertical distance between the undisturbed free-surface and the surface of the entering body is important but not the surfaces on their own. If this is valid within the second-order theory, then formula (97) can be compared with the experimental results by Cointe and Armand [11] for a circular cylinder entering a liquid at constant velocity. Note that a cylinder can be approximated by a parabolic contour only close to the impact point. The non-dimensional hydrodynamic force (97) is shown in Fig. 4 by the solid curve and the experimental results from [11] by points. It is seen that formula (97) corresponds well to the experimental results and that the second-order force contribution significantly improves the Wagner theory.

6 Conclusion

In this paper the simplest problem of water impact was considered. The problem is simple because the rigid surface is flat, the initial shape of the free-surface is parabolic and the impact velocity is constant. More-complex shapes and different impact conditions would require a more sophisticated higher-order impact theory. However, the present simple analysis is helpful as a reference for more general and more accurate impact theories (see [12], where a two-dimensional entry problem is studied in a higher-order approximation). This is why we tried to give as many analytical details as possible.

Note that stretched coordinates were used in the present analysis. Stretched coordinates are helpful because they allow us to fix the dimension of the contact region and perform a formal asymptotic analysis.

It is interesting to notice that the obtained second-order velocity potential is rather complicated but the second-order pressure distribution in the contact region has the same form as in the first-order theory by Wagner. One may expect that this result is valid also in the three-dimensional case.

It is important to notice that the second-order impact theory does not give a contribution to the dimension of the contact region as a function of time. This is in agreement with the well-known observation that the first-order Wagner theory predicts the dimension of the contact region reasonably well, even if the first-order prediction of the hydrodynamic force is not satisfactory.

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Appendix

In this Appendix the solution of the boundary-value problem (44)–(47) is derived and the formulae (48) and (49) are obtained.

In order to obtain the solution, we introduce the complex potential $w_1(z) = \phi_1 + \mathrm{i}\psi_1$, which is analytical in the lower half-plane $\eta < 0$ and satisfies the mixed boundary conditions

$$\psi_1 = 0 \quad (|\xi| < 1,\ \eta = 0), \tag{98}$$

$$\phi_1 = \frac{1}{8}\left[1 - \xi^2 \log\left(1 - \frac{1}{\xi^2}\right)\right] \quad (|\xi| > 1,\ \eta = 0). \tag{99}$$

The complex potential is sought in the form

$$w_1(z) = \frac{1}{8}\left[1 - z^2 \log\left(1 - \frac{1}{z^2}\right)\right] + \hat{w}_1(z), \tag{100}$$

where the branch of the multi-valued function $\log(1 - 1/z^2)$ is defined on the plane z with a cut along the interval $\eta = 0, -1 < \xi < 1$ and is chosen in such a way that

$$\log\left(1 - \frac{1}{z^2}\right) = \log\left|1 - \frac{1}{z^2}\right| + \mathrm{i}A(z),$$
$$A(z) = 0 \quad (|\xi| > 1,\ \eta = 0),$$
$$A(z) = -\pi\,\mathrm{sgn}(\xi) \quad (|\xi| < 1,\ \eta = -0). \tag{101}$$

Equations 99–101 give that

$$\mathfrak{Re}[\hat{w}_1(\xi - \mathrm{i}0)] = 0 \quad (|\xi| > 1). \tag{102}$$

Correspondingly, the body boundary condition (98) and (100)–(101) provide

$$\mathfrak{Im}[\hat{w}_1(\xi - \mathrm{i}0)] = -\frac{\pi}{8}\xi|\xi| \quad (|\xi| < 1). \tag{103}$$

The auxiliary function $V(z) = \hat{w}_1(z)\sqrt{z^2 - 1}$, where the branch of the function $\sqrt{z^2 - 1}$ is defined by (30), satisfies the boundary conditions

$$\mathfrak{Re}[V(\xi - \mathrm{i}0)] = 0 \quad (|\xi| > 1), \tag{104}$$

$$\mathfrak{Re}[V(\xi - \mathrm{i}0)] = -\frac{\pi}{8}\xi|\xi|\sqrt{1 - \xi^2} \quad (|\xi| < 1). \tag{105}$$

The analytic function $V(z)$ in the lower half-plane $\eta < 0$ is sought in the form of the Cauchy integral

$$V(z) = \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\tau(\sigma)\mathrm{d}\sigma}{\sigma - z} + \mathrm{i}V_1, \tag{106}$$

where V_1 is an arbitrary real constant. By using the Plemely's formula at the limit $\eta \to -0$, we obtain

$$V(\xi - \mathrm{i}0) = -\frac{1}{2}\tau(\xi) + \frac{1}{2\pi \mathrm{i}}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\tau(\sigma)\mathrm{d}\sigma}{\sigma - \xi} + \mathrm{i}V_1, \tag{107}$$

where p.v. stands for principal value integral. Comparing (107) and (104), we find

$$\tau(\xi) = \frac{\pi}{4}\xi|\xi|\sqrt{1 - \xi^2} \quad (|\xi| < 1), \quad \tau(\xi) = 0 \quad (|\xi| > 1). \tag{108}$$

Equations 100, 101 give the second-order velocity potential in the contact region $\eta = -0, -1 < \xi < 1$ as

$$\phi_1(\xi, -0) = \frac{1}{8}\left[1 - \xi^2 \log\left(1 - \frac{1}{\xi^2}\right)\right] - \frac{1}{\sqrt{1 - \xi^2}}\mathfrak{Im}[V(\xi - \mathrm{i}0)]. \tag{109}$$

On the free-surface, $\xi > 1$, we find

$$\psi_1(\xi, 0) = \mathfrak{Im}[\hat{w}_1(\xi - \mathrm{i}0)] = \frac{1}{\sqrt{\xi^2 - 1}} \mathfrak{Im}[V(\xi - \mathrm{i}0)]. \tag{110}$$

It is seen that, to obtain Eqs. 48 and 49, one needs to evaluate the imaginary part of the analytic function $V(z)$ along the liquid boundary $z = \xi - \mathrm{i}0$. Equations 107 and 108 give

$$\mathfrak{Im}[V(\xi - \mathrm{i}0)] = -\frac{1}{2\pi} \cdot \frac{\pi}{4} \text{ p.v.} \int_{-1}^{1} \frac{\sigma|\sigma|\sqrt{1-\sigma^2}\mathrm{d}\sigma}{\sigma - \xi} + V_1. \tag{111}$$

Some algebra gives

$$\mathfrak{Im}[V(\xi - \mathrm{i}0)] = \frac{1}{4} \int_0^1 \frac{\sigma^2(1-\sigma^2)\mathrm{d}\sigma}{\sigma^2 + \xi^2 - 1} + V_1 \tag{112}$$

and

$$\mathfrak{Im}[V(\xi - \mathrm{i}0)] = \frac{1}{4}\left[-\frac{1}{3} + \xi^2 + \xi^2(1-\xi^2)\int_0^1 \frac{\mathrm{d}\sigma}{\sigma^2 + \xi^2 - 1}\right] + V_1. \tag{113}$$

The value of the integral in (113) is dependent on the value of the variable ξ. In the contact region, $|\xi| < 1$ and $\xi^2 - 1 = -a^2$, which yields

$$\int_0^1 \frac{\mathrm{d}\sigma}{\sigma^2 + \xi^2 - 1} = \frac{\log|\xi| - \log(1 + \sqrt{1-\xi^2})}{\sqrt{1-\xi^2}}$$

and finally

$$\mathfrak{Im}[V(\xi - \mathrm{i}0)] = \frac{\xi^2 - 1/3}{4} + \frac{1}{4}\xi^2\sqrt{1-\xi^2}\left[\log|\xi| - \log(1 + \sqrt{1-\xi^2})\right] + V_1 \quad (|\xi| < 1). \tag{114}$$

On the free-surface, $|\xi| > 1$ and $\xi^2 - 1 = a^2$, which yields

$$\int_0^1 \frac{\mathrm{d}\sigma}{\sigma^2 + \xi^2 - 1} = \frac{1}{\sqrt{1-\xi^2}} \arctan \frac{1}{\sqrt{1-\xi^2}} \tag{115}$$

and finally

$$\mathfrak{Im}[V(\xi - \mathrm{i}0)] = \frac{\xi^2 - 1/3}{4} - \frac{1}{4}\xi^2\sqrt{\xi^2 - 1} \arctan \frac{1}{\sqrt{1-\xi^2}} + V_1. \tag{116}$$

Equations 109 and 114 provide the distribution of the second-order velocity potential in the contact region as

$$\phi_1(\xi, 0) = \frac{1}{8} - \frac{1}{8}\xi^2 \log(1-\xi^2) + \frac{1}{4}\sqrt{1-\xi^2} + \frac{1}{4}\xi^2 \log\left(1 + \sqrt{1-\xi^2}\right) - \frac{V_2}{\sqrt{1-\xi^2}}, \quad V_2 = V_1 + \frac{1}{6} \tag{117}$$

and the stream function on the free-surface as

$$\psi_1(\xi, 0) = \frac{1}{4}\sqrt{\xi^2 - 1} - \frac{1}{4}\xi^2 \arctan \frac{1}{\sqrt{1-\xi^2}} + \frac{V_2}{\sqrt{\xi^2 - 1}} \quad (\xi > 1). \tag{118}$$

By using the equality $\phi_{1\eta} = -\psi_{1\xi}$, we obtain the second-order vertical velocity on the free-surface, $\xi > 1$, in the form

$$\frac{\partial \phi_1}{\partial \eta}(\xi, 0) = \frac{1}{2}\xi \left\{ \arctan\left[\frac{1}{\sqrt{\xi^2 - 1}}\right] - \frac{1}{\sqrt{\xi^2 - 1}} + \frac{2V_2}{(\xi^2 - 1)^{\frac{3}{2}}} \right\}. \tag{119}$$

The terms with V_2 in (117)–(119) are the most singular ones. The matching of the second-order outer solution with the inner solution (see Sect. 4) gives $V_2 = 0$ and turns (119) into (48) and (117) into (49).

The second-order horizontal velocity in the contact region is obtained as

$$\phi_{1\xi}(\xi,0) = -\frac{1}{4}\xi\log(1-\xi^2) + \frac{1}{2}\xi\log\left(1+\sqrt{1-\xi^2}\right) + \frac{1}{4}\frac{\xi}{1-\xi^2} - \frac{\xi}{2}\frac{1}{\sqrt{1-\xi^2}},$$

which gives

$$\xi\phi_{1\xi}(\xi,0) - 2\phi_1(\xi,0) = \frac{1}{4}\frac{\xi^2}{1-\xi^2} - \frac{1}{4} - \frac{1}{2}\frac{\xi^2}{\sqrt{1-\xi^2}} - \frac{1}{2}\sqrt{1-\xi^2}.$$

The latter formula is used for deriving Eq. 53.

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